Failure Models Indexed by Time and Usage

by

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Abstract

All commonly used failure models, both univariate and multivariate, are indexed by a single scale, typically time. This thesis considers the case of indexing a failure model by two scales, namely time and usage. A family of models that is indexed by these two scales of measurement, and attempts to capture the dependence between them, is developed and an example given in detail. Extensions to this basic model, and problems associated with them, are explored. As an aside, an idealised stochastic process for modelling usage is introduced.

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1 Introduction

A glance at any book or journal in the field of reliability and survival analysis will show that much time and effort has been devoted to probabilistic models for failure of an item or system of items. One can choose to model the time to failure of an object from a wide variety of univariate probability distributions such as the exponential, Weibull or gamma. The times to failure of a system of units can be modelled by one of many multivariate distributions; see Hougaard (1987) for a review of some of the more commonly used. All these distributions have a feature in common, namely that they are indexed on a single variable or scale of measurement. Often this variable is time but may be number of cycles, amount of exposure to a hazardous material; any variable that the modeller thinks relevant in fact.

Whilst these models are widely used and are very versatile, there are certain situations where one would like failure of an item to be indexed by more than one variable. In this case the existing models are not satisfactory. A classic example is in the area of automobile warranties, where a manufacturer will agree to repair or replace a vehicle if it fails within, say, 3 years or 30,000 miles from production. If a manufacturer wishes to investigate the expected number of warranty claims or tries to find a warranty that maximises his profit margin, then it is necessary to have a failure model for automobiles that is in terms of time to failure and number of miles to failure. Another example would be measuring the number of deaths amongst mineworkers in terms of age at death and amount of exposure to hazardous dust whilst at work. In both these examples the two scales of measurement are obviously dependent - what is needed is a failure model indexed jointly by the two scales. In this thesis we assume these two scales are time and usage, but this work could equally be applied to time and some other index like exposure to hazardous dust.

To the best of my knowledge almost no work has been published that looks at this situation. That failure can be represented by more than one scale of measurement has been recognised, see Farewell and Cox (1979) or Oakes (1988). In these papers the authors consider the problem of reducing several scales of measurement down to one that is most informative in some sense, with the starting scales becoming time-dependent covariates.

The innovative nature of the idea here must be emphasised; this topic has received almost no attention in the literature despite the fact that such models arise naturally in many situations of interest. A naive solution to the problem of having a suitable model would be to select one of the available bivariate failure models and then attempt to justify its use as a time/use failure model in some manner. However, this is not the approach taken in this thesis; instead, assumptions are made concerning the dependence between the two scales of measurement and a model is derived from those assumptions. It is hoped that the result is a model with more justification than one developed from the former approach, which would employ a model that was really designed to describe the joint failure of two units in terms of one scale rather than the failure of one unit in terms of two scales.

This thesis is split into 5 further sections. Section 2 introduces a family of models indexed by time and usage then section 3 considers an example of this model in some detail. Section 4 considers generalisations of the model and some problems that may be encountered with them, whilst section 5 looks at a more realistic stochastic process to model usage. Section 6 concludes the thesis and outlines possible areas for future research in this topic.

2 A bivariate failure model indexed by time and usage.

The aim of this section is to introduce a generic bivariate model for failure in terms of both time and usage.

Consider some unit or object like a car, a light bulb or any item which can accumulate use; in the case of an automobile this use will be mileage, in the case of a light bulb it may be number of times switched on or the amount of time that it is lit. To begin with, we observe that this unit may age over time, even though it is not used, due to rusting, humidity, adverse temperature and other environmental effects. These effects are dynamic, that is change over time. Singpurwalla and Youngren (1990) have shown that if $\omega(t)$ denotes the failure rate of the unit under ideal (or laboratory) storage conditions, and if the dynamics of the storage environment are described by a stochastic process known as a "gamma process" with parameters $a(t)$ and b, then the failure rate of the unit under storage is

$$r(t) = a(t) \ \ln\left(1 + \frac{\omega(t)}{b} \right) \tag{2.1}$$

If $a(t) = a$ and $\omega(t) = \omega$ then $r(t)$ is a constant and so ageing under storage would be that provided by an exponential distribution. Similar results are obtained when the gamma process is replaced by a "shot-noise process".

From (2.1) we have a failure rate for the unit when it is not used. Of course one can choose any failure rate under non-use that one likes; the point of (2.1) is to demonstrate that models have been developed in the literature to meet this requirement. Now let us put the unit to use, and denote the amount of usage at time t as $M(t)$. We use the notion of proportional hazard (Cox (1972)) and suppose that the effect of usage is to increase the conditional failure rate $r(t|M(t))$ of the unit from $r(t)$ to say $r(t) + \beta M(t)$, where $\beta \geq 0$ is a constant; other forms are possible. We are uncertain about the usage at time t and so $M(t)$ is assumed to be some positive and non-decreasing stochastic process, in particular a Poisson process with intensity function $\lambda(t)$. Intuitively $\lambda(t)$ measures the rate of use of the item over time, and figure 1 gives three possible

functions that are candidates for $\lambda(t)$: respectively a constant rate of use, cyclical use and a rate of use that decreases as the unit becomes old. Observe that this makes $M(t)$ a discrete quantity taking values in the non-negative integers, although one is at liberty to consider other, possibly real-valued, non-decreasing stochastic processes to model $M(t)$.

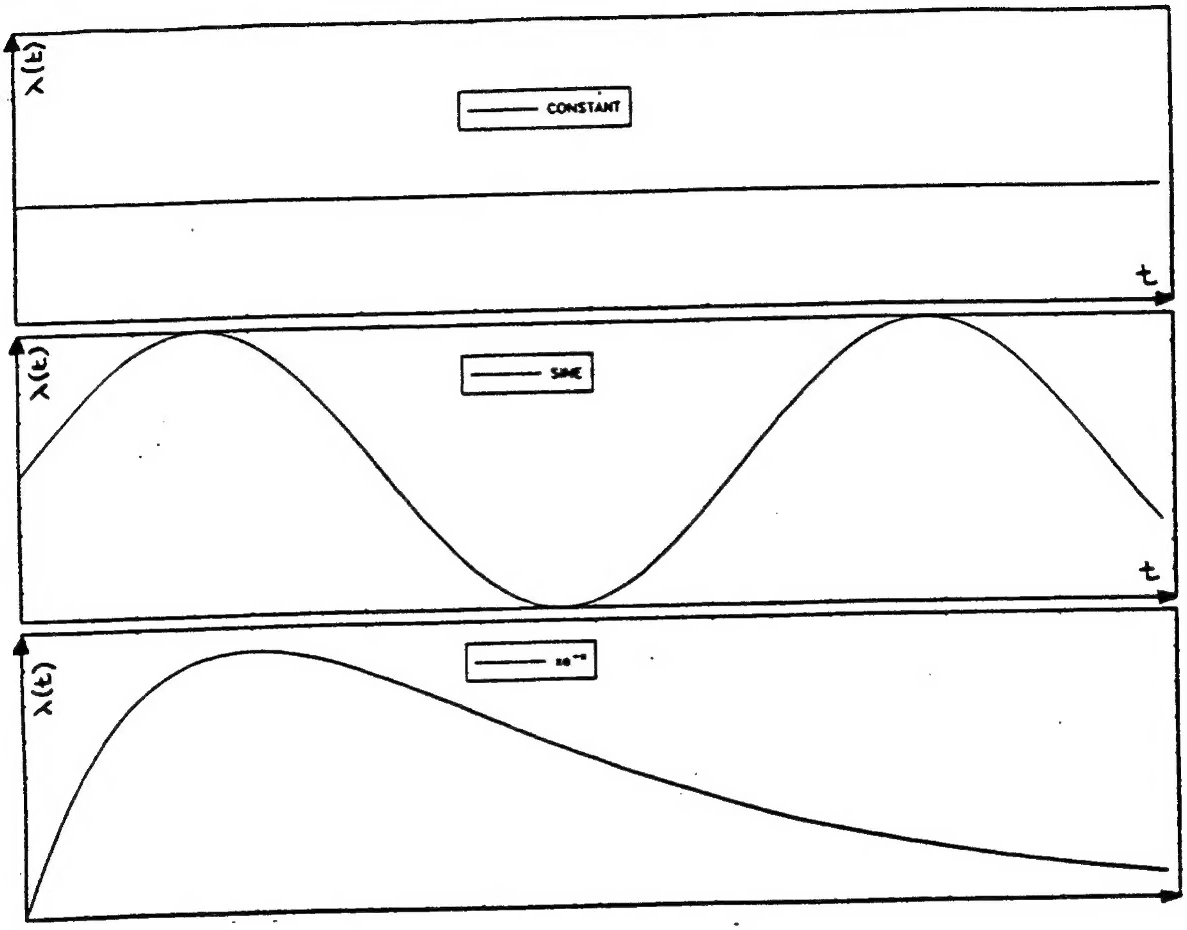

Figure 1 Three candidate functions for $\lambda(t)$

With these assumptions we have tried to model the physics of the relationship between time and usage to failure in a simple but intuitive way. In any event, we are now able to derive a bivariate density for failure of the unit in terms of time and use based on these assumptions. The derivation is quite straightforward, and will only be outlined below; readers interested in the full technical details of the derivation are referred to the appendix.

Let T denote time to failure and U usage to failure. The first step is to reason as follows

$$
\begin{aligned}
(2.2)\quad f_{T,U}(t, u) &= \text{density of failure at time t and use u} \\
&= \text{density of failure at time t and with use at time t of u} \\
&= f_{T,M(t)}(t, u) \\
&= f_{T|M(t)}(t|\ u)\ f_{M(t)}(u) \\
&= f_{T|M(t)}(t|\ u)\ \frac{(\Lambda(t))^u}{u!}\ e^{-\Lambda(t)}
\end{aligned}
$$

where $\Lambda(t) = \int_0^t \lambda(s)\, ds$, since $M(t)$ is a Poisson process.

Using the well known identity connecting any continuous probability density $f(t)$ with its failure rate $h(t)$

$$f(t) = h(t) \exp\left(-\int_0^t h(s)\, ds \right)$$

we can say

$$(2.3)\quad f_{T|M(t)}(t|\ M(t)=u) = (r(t)+\beta u) \times \exp\left(-\int_0^t (r(s)+\beta M(s))\, ds \mid M(t)=u \right)$$

where $\int f(s)\, ds \mid f(t)=u$ denotes the integral of $f(s)$ given that $f(t)=u$.

By using this expression, we can follow the method outlined by Lemoine and Wenocur (1986) and arrive at

$$(2.4)\qquad f_{T|M(t)}(t|\ u) = \frac{(r(t) + \beta u)\, e^{-R(t)}}{(\ \Lambda(t)\ )^u} \left(\int_0^t \lambda(s)\, e^{-\beta(t-s)}\ ds \right)^u$$

where $R(t) = \int_0^t r(s)\, ds$, the cumulative hazard.

Now incorporate (2.4) into (2.2) to obtain

$$(2.5) \qquad f_{T,U}(t, u) = \frac{(r(t) + \beta u)\, e^{-R(t)}\; e^{-\Lambda(t)}}{u!} \left(\int_0^t \lambda(s)\, e^{-\beta(t-s)}\, ds \right)^u$$

where $t \geq 0$, $u = 0, 1, 2, \ldots$

It must be emphasised that this is not a bivariate density for failure in the usual sense of modelling the joint failure of two items, but rather describes the failure of a single item in terms of time and use. There is flexibility in this model, as one is at liberty to assign any intensity of use function $\lambda(t)$ and failure rate for storage $r(t)$ that seems plausible, and indeed to alter the relationship $r(t)+\beta M(t)$ between them. Note that the integral appearing in (2.5) is the convolution of $\lambda(s)$ with $e^{\beta s}$, so can be found via Laplace transforms if direct integration proves troublesome.

The above density enjoys some analytic tractability, even in the most general case. For instance, the marginal of T is

$$(2.6) \quad f_T(t) = \left(r(t)+\beta\int_0^t \lambda(s)\, e^{-\beta(t-s)}\, ds\right) \times \exp\left\{-R(t) - \int_0^t \lambda(s)\left(1 - e^{-\beta(t-s)}\right) ds\right\}$$

$t \geq 0$.

This gives a marginal survival of

$$(2.7) \qquad \bar{F}(t) = P(T>t) = \exp\left\{ -R(t) - \int_0^t \lambda(s)\left(1 - e^{-\beta(t-s)}\right) ds \right\} \qquad t \geq 0.$$

It is then an easy matter to find the marginal failure rate as

$$(2.8) \qquad r_T(t) = \frac{f_T(t)}{\bar{F}(t)} = r(t) + \beta\int_0^t \lambda(s)\, e^{-\beta(t-s)}\, ds \qquad t \geq 0\,.$$

Equations (2.6), (2.7) and (2.8) defines a probability for time to failure that considers both environmental and usage stresses on a unit, and as such is of merit by itself as a univariate model.

Given (2.5) and (2.6) we have the conditional density of usage to failure given time to failure as

$$(2.9) \quad P(U{=}u \mid T{=}t) \;=\; \frac{r(t) + \beta u}{r(t) + \beta\eta(t)} \times \frac{(\eta(t))^u}{u!}\; e^{-\eta(t)} \qquad u = 0, 1, 2, \ldots$$

$$\text{where } \eta(t) = \int_0^t \lambda(s)\, e^{-\beta(t-s)}\, ds$$

Note that this is a different density from M(t), usage at time t, unless $\beta=0$. Whereas the expected value of M(t) is $\Lambda(t)$, the expected value of U given T=t can be shown to be

$$(2.10) \qquad E(U \mid T{=}t\,) = \eta(t) \times \left(1 + \frac{\beta}{r(t) + \beta\eta(t)} \right).$$

which is seen to be equal to $\Lambda(t)$ if and only if $\beta=0$. In general, the marginal of U is difficult to calculate, although as we see in Section 3 it can often be obtained in special cases.

Uncertainty about the parameter β might be described by placing a prior distribution on it. Uncertainty about the functions r(t) and $\lambda(t)$ could be described by making them stochastic processes; thus M(t) would become a doubly stochastic or conditional Poisson process. Section 3 looks at these ideas for a particular example of the model.

3 Example

In this example we consider the simple situation of the failure of an electrical switch in terms of T, the time to failure and U, the number of times the switch is used. We assume that under storage conditions the switch ages according to an exponential distribution with rate α ($r(t) = \alpha$ in the notation of section 2). Each time the switch is used this failure rate is incremented by an amount β because of the extra stress induced upon it (so $r(t|\ M(t)) = \alpha+\beta M(t)$ in the notation of section 2). Finally, we assume that use of the switch occurs as a homogeneous Poisson process (that is to say the time between uses of the switch are independent and identically distributed exponential random variables) with rate λ. Under these assumptions we may apply our model (2.5) and obtain the density for failure at time t and number of uses u as

$$(3.1)\quad f_{T,U}(t,u) = \frac{(\alpha+\beta u)\, e^{-(\alpha+\lambda)t}}{u!}\left(\frac{\lambda}{\beta}\left(1 - e^{-\beta t}\right)\right)^{u}; \quad t \geq 0\,, \quad u = 0, 1, 2, \ldots$$

The density is continuous in t but only defined for u an integer, so consists of a sequence of univariate functions, continuous in t, equally spaced along the U-axis at u=0, 1, 2, ... The plots of such a density will be referred to as *bivariate density sheets.*

The formulae given by (2.6) to (2.10) can be applied by letting $r(t) = \alpha$ and $\lambda(t) = \lambda$; for example the marginal survival to time t, and marginal failure rate, of the switch are

$$(3.2)\qquad \begin{aligned} \bar{F}_T(t) &= \exp\left\{ -(\alpha + \lambda)t + \frac{\lambda}{\beta}(1 - e^{-\beta t})\right\} \\ r_T(t) &= \alpha + \lambda\,(1 - e^{-\beta t})\,, \qquad\qquad t \geq 0\,. \end{aligned}$$

In this special case we can exploit the homogeneity of the Poisson process to obtain the marginal for use to failure:

$$(3.3)\qquad P(U = u) = \frac{(\alpha + \beta u)\, \lambda^{u}}{\prod_{i=0}^{u} (\lambda + \alpha + \beta i)} \qquad u = 0, 1, 2, \ldots$$

We also have an expression for the distribution function:

$$(3.4)\quad P(T \le t,\ U \le u) = \sum_{n=0}^{u} \frac{\lambda^n(\alpha+\beta n)}{\beta^n\, n!} \left\{ \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(1-\exp\{-(\lambda+\alpha+\beta i)t\})}{\lambda+\alpha+\beta i} \right\}$$

Figure 2 is a plot of the density (3.1) when α=0.2, β=0.2 and λ=1.5.

The marginal moments for T and U can also be written down in this case, although only rather messily in terms of the incomplete gamma function $\gamma(a, n)$.

Let $x = \lambda/\beta$ and $y = \alpha/\beta$. Then

$$(3.4)\quad E(T) = \frac{\exp(x)}{\beta}\ x^{-(x+y)} \times \gamma(x+y, x)$$

and

$$(3.5)\quad E(U) = \exp(x)\, x^{-(x+y)} \times \Big(\gamma(2+x+y, x) - (1+2x+y)\ \gamma(1+x+y, x) + x(1+x+y)\ \gamma(x+y, x) \Big).$$

Now we consider the placing of priors on the parameters (α, β, λ) of (3.1), and address the question of using data to update our prior distributions. The first observation is that if α and β are independent of λ then the gamma distribution is a conjugate prior for λ. This is easy to see from (3.1) by noting that

$$\begin{aligned} L(\lambda \mid t, u) &= f(t, u \mid \lambda) \\ &\propto \lambda^u e^{-t\lambda} \end{aligned}$$

Suppose λ has a gamma distribution with shape ν_1 and scale θ_1. Given n (independent) observed time-use failure pairs $(t_1, u_1), \ldots, (t_n, u_n)$ we have a posterior for λ :

$$(3.5)\quad \pi(\lambda \mid (t_1, u_1), \ldots, (t_n, u_n)) \propto \pi(\lambda) \times L(\lambda \mid (t_1, u_1), \ldots, (t_n, u_n))$$

$$\propto \lambda^{\nu_1 - 1 + \sum u_i} e^{-(\theta_1 + \sum t_i)\lambda}$$

so that the posterior of λ is gamma with shape $\nu_1 + \sum u_i$ and scale $\theta_1 + \sum t_i$.

The likelihoods for α and β do not appear to possess a recognisable conjugate prior. However, in the case of α a posterior can still be written down in closed form, albeit an increasingly complex one as we increase the amount of data, if we assume a gamma prior on α. Assume α is independent λ. The likelihood of α can be written :

$$L(\alpha \mid \beta, t, u) \propto (\alpha + \beta u)\, e^{-\alpha t}$$

Assume that the prior on α is a gamma with shape ν_2 and scale θ_2. Then the posterior of α is of the form

$$\text{(3.6)} \quad \pi(\alpha \mid \beta, (t_1,u_1),\ldots,(t_n,u_n)) \propto \pi(\alpha) \times L(\alpha \mid \beta, (t_1,u_1),\ldots,(t_n,u_n))$$
$$\propto e^{-(\theta_2 + \sum t_i)\,\alpha} \times \alpha^{\nu_2 - 1} \prod_{i=1}^{n} (\alpha + \beta u_i)$$

The normalising constant here will be

$$\text{(3.7)} \quad \int_0^\infty e^{-(\theta_2 + \sum t_i)\,\alpha} \times \alpha^{\nu_2 - 1} \prod_{i=1}^{n} (\alpha + \beta u_i)\; d\alpha$$

Expanding the product, we see that the integrand is a finite sum of powers of α multiplied by $\exp\{-(\theta_2 + \sum t_i)\alpha\}$, so (3.7), the normalising constant for the posterior of α given by (3.6), is just a finite sum of gamma functions. Thus we can give the exact posterior of α. This posterior is dependent upon β, so even if we assume α and β to be a priori independent, they are dependent in the posterior.

Similar calculations for β are less straightforward, due to the more complex nature of the likelihood. There is no closed form posterior for β for any of the commonly used priors, and it appears that one would have to employ simulation or some other approximation to obtain a posterior. If β were known or assumed

as some value, then (3.5) and (3.6) show that one could proceed to calculate exact posteriors for α and λ given that they had independent gamma priors. The independence of α and λ would remain, so these calculations could be done separately. If some prior was placed on β, then we would need to calculate a joint posterior for α and β because of the dependence that is introduced between them.

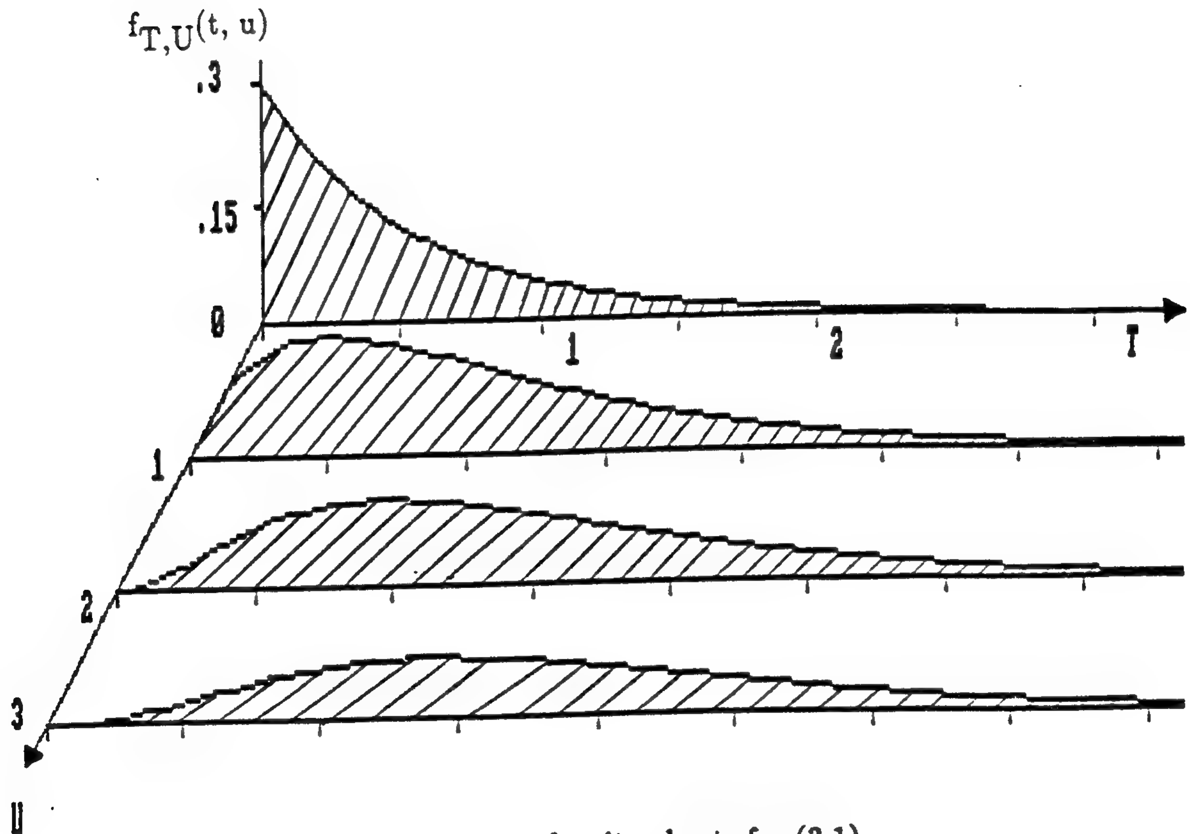

Figure 2 Bivariate density sheets for (3.1)

4 Using other stochastic processes to model usage

There are several ways by which the model outlined in Section 2 can be generalised. One that comes straight to mind is to model cumulative usage as some stochastic process other than Poisson, perhaps some process taking values in $[0,\infty)$ instead of the non-negative integers. There are numerous stochastic processes that one could look at, and intuitively any process M(t) that satisfies

i) $M(0) = 0$
ii) $M(t) \geq 0 \quad \forall t$
iii) M(t) is non-decreasing

is a possible candidate to model usage.

Given that we decide on some process M(t), then from (2.2) and (2.3) we see that to obtain a density for failure in time and usage requires us to know two things about M(t)

a) The density of M(t) and

b) an evaluation of $\exp\left(-\beta\int_0^t M(s)\,ds \mid M(t)=u\right)$

Recall that we assume the failure rate of a unit when it is not used is r(t). The effect of usage m is to increase the failure rate to $r(t)+\beta m$, where β is a constant. In other words :

$$r(t \mid m) = r(t)+\beta m$$

Since usage is assumed to be a stochastic process M(t), we rewrite the conditional failure rate as:

$$r(t \mid M(t)) = r(t)+\beta M(t)$$

What we desire is the density of failure at time t and usage u. Let T be the

random variable denoting time to failure, and let U be the random variable denoting usage to failure. As in section 2, we reason as follows:

$$
\begin{aligned}
(4.1) \qquad f(t, u) &= \text{density of failure at time } t \text{ and usage } u \\
&= \text{density of failure at time } t \text{ and with usage at time } t \text{ of } u \\
&= f(t, M(t)=u) \\
&= f(M(t)) \times f(t \,|M(t))
\end{aligned}
$$

Using our conditional failure rate $r(t|\, M(t))$ we can say

$$
\begin{aligned}
(4.2) \qquad f(t|\, M(t)=u) &= (r(t)+\beta u) \times \exp\left(-\int_0^t r(t)+\beta M(s)\, ds \mid M(t)=u \right) \\
&= (r(t)+\beta u)\, e^{-R(t)} \times \exp\left(-\beta \int_0^t M(s)\, ds \mid M(t)=u \right) \\
&\quad \text{where } R(t) = \int_0^t r(s)\, ds
\end{aligned}
$$

Combining equations (4.1) and (4.2) we see that the required joint density is of the form

$$
(4.3) \quad f(t, u) = f(M(t)=u) \times (r(t)+\beta u)\, e^{-R(t)} \exp\left(-\beta \int_0^t M(s)\, ds \mid M(t)=u \right).
$$

(4.3) provides a most general expression for a class of densities indexed by time and usage. By making M(t) a Poisson process, we obtain the model of section 2.

What about M(t) as another stochastic process? I have studied, as a first example, the case of modelling M(t) as a homogeneous compound Poisson process. This can be described as a homogeneous Poisson process where the jump at each event is no longer 1, but some random variable. I considered the case where the jump sizes are independent and identically distributed exponential random variables. In other words, if N(t) denotes the number of events at time t (which is a homogeneous Poisson process) and U_i are iid exponential then

$$
(4.4) \qquad M(t) = \begin{cases} 0, & \text{if } N(t)=0 \\ \sum_{i=1}^{N(t)} U_i\,, & \text{if } N(t) \geq 1 \end{cases}
$$

This process satisfies the three criteria i), ii) and iii). A typical realisation of this process is given by figure 3. M(t) can now take any non-negative value. To give this model some practical explanation, imagine that M(t) is the number of miles

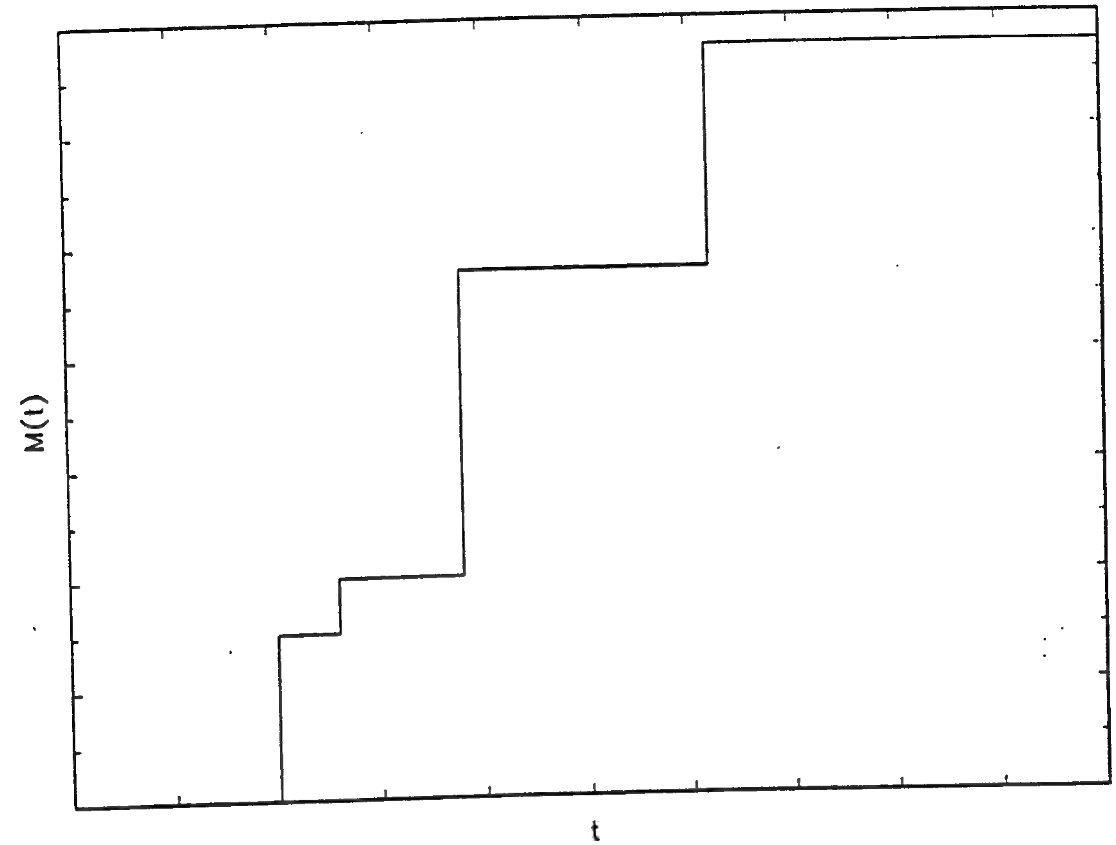

Figure 3 Typical realisation of a compound Poisson process M(t)

an automobile has driven. Under this model we are then assuming that the car is used according to a homogeneous Poisson process, and that each time it is used the number of miles driven is an exponential random variable. Amongst other things, this means that we assume each time between uses of the vehicle is independent. Since this process is a 'jump' process, we are also tacitly assuming that these miles are accumulated instantly, whereas of course in reality they are accumulated over some period of time. If we assume, as we did in the example of Section 3, that the conditional failure rate is $\alpha + \beta M(t)$, then we are ready to derive the joint density via the same method employed in Section 2. M(t) is compound Poisson; let the usage events occur as a Poisson process with rate λ

and let each jump in the process be exponential with mean 1. The density of M(t) can be computed; it is continuous on $(0,\infty)$ and has positive probability mass at 0, since there is always a positive probability that N(t)=0.

The density of M(t) is given by:

$$(4.5)\quad P(M(t) = 0) = e^{-\lambda t} \quad ; \quad f_{M(t)}(u) = e^{-u-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n\, u^{n-1}}{n!\,(n-1)!} \qquad u > 0$$

It remains to calculate $f_{T|M(t)}(t|\, u)$. Observe from (2.3) that this is

$$(4.6)\quad f_{T|M(t)}(t|\, u) \;=\; (\alpha+\beta u)\, e^{-\alpha t} \times \exp\left\{ -\beta \int_0^t M(s)\, ds \mid M(t)=u \right\}$$

To find $\int_0^t M(s)\, ds$, the area under M(s) between 0 and t, given that M(t)=u was quite straightforward in section 2 when M(s) was Poisson but is now much more difficult. I have been unable to obtain a closed form for (4.3) when M(t) is compound Poisson, but by conditioning on N(t) and combining with (4.5) I can say that for $t \geq 0$ and u>0 there is a joint continuous density of the form

$$(4.7)\quad f_{T,U}(t,u) = (\alpha + \beta u)\, e^{-(\alpha+\lambda)t}\, e^{-u}\, e^{-\beta t u} \times \sum_{n=1}^{\infty} \lambda^n\; I_n(t,\, u;\, \beta)$$

where $I_n(t,\, u;\, \beta)$ is a 2n-dimensional integral given by

$$I_n(t,\, u;\, \beta) = \int_{\Omega_n} \exp\left\{ \beta \sum_{i=1}^{n} s_i\, u_i \right\}\, d(\underline{s},\, \underline{u})$$

$$\text{and}\quad \Omega_n = \left\{ (\underline{s},\underline{u}) \in \mathbf{R}^{2n} \mid\; u_i \geq 0 \;\; ; \;\; \sum_{i=1}^{n} u_i = u \;\; ; \;\; 0 \leq s_1 \leq \dots \leq s_n \leq t \right\}$$

As with (2.5), the details of the derivation of (4.7) are rather cumbersome but present few difficulties and have been relegated to the appendix. This integral has finite value, and of course can be evaluated numerically. It is possible to analytically integrate out $(s_1,\dots,s_n)$, leaving an n-dimensional integral over $(u_1,\dots,u_n)$ to calculate numerically. We are integrating over the simplex $\sum u_i = u$, and so can reduce this integral to an (n−1)-dimensional problem. This is still a rather forbidding task for even moderate values of n, and to become practicable further simplification of (4.4) will be necessary.

As there is positive probability mass at M(t)=0, so there is mass at U=0 (that is, along the T-axis) in the joint density. We can show that

$$(4.8) \qquad P(U=0) = \frac{\alpha}{\alpha + \lambda} \ .$$

It is distributed along this axis with mass function

$$f_{T,U}(t, 0) = \alpha\, e^{-(\alpha+\lambda)t} \qquad t \geq 0$$

Although the joint density appears to be beyond a closed form representation, by following the method of Lemoine and Wenocur (1986) one can obtain the marginal survival for time to failure as :

$$(4.9) \qquad P(T>t) = (1+\beta t)^{\frac{\lambda}{\beta}}\, e^{-(\alpha+\lambda)t} \qquad t \geq 0$$

The marginal failure rate is

$$(4.10) \qquad r_T(t) = \alpha + \lambda \frac{\beta t}{1+\beta t} \ . \qquad t \geq 0$$

We notice a similarity between (4.10) and the marginal failure rate of the model example in section 3, given by (3.2). Recall that the model there simply assumed M(t) as a homogeneous Poisson process with intensity λ. In both cases we have :

$$r_T(0) = \alpha \ ; \qquad \lim_{t\to\infty} r_T(t) = \alpha+\lambda \ ; \qquad r_T(t) \text{ is strictly increasing.}$$

The density specified in (4.7) is more easily approximated via simulation. The integral in (4.6) can be easily simulated by computer and this enables us to approximate a density value for any given t and u. If no further simplification of (4.7) is possible, then numerical simulation is the easier and quicker option over numerical integration. Details of a simulation plan are to be found in the appendix. Figure 4 overpage is a plot of the simulated joint density for $\alpha=\beta=\lambda=1$.

In conclusion of this section, it is clear that modelling usage by a real-valued stochastic process, instead of a Poisson process, may make matters more realistic but does increase the complexity of the resulting density. The situation examined in this section is a good example of what can happen; the compound Poisson process is one of the more analytically tractable real-valued stochastic processes yet it is still difficult to define the associated joint density of failure.

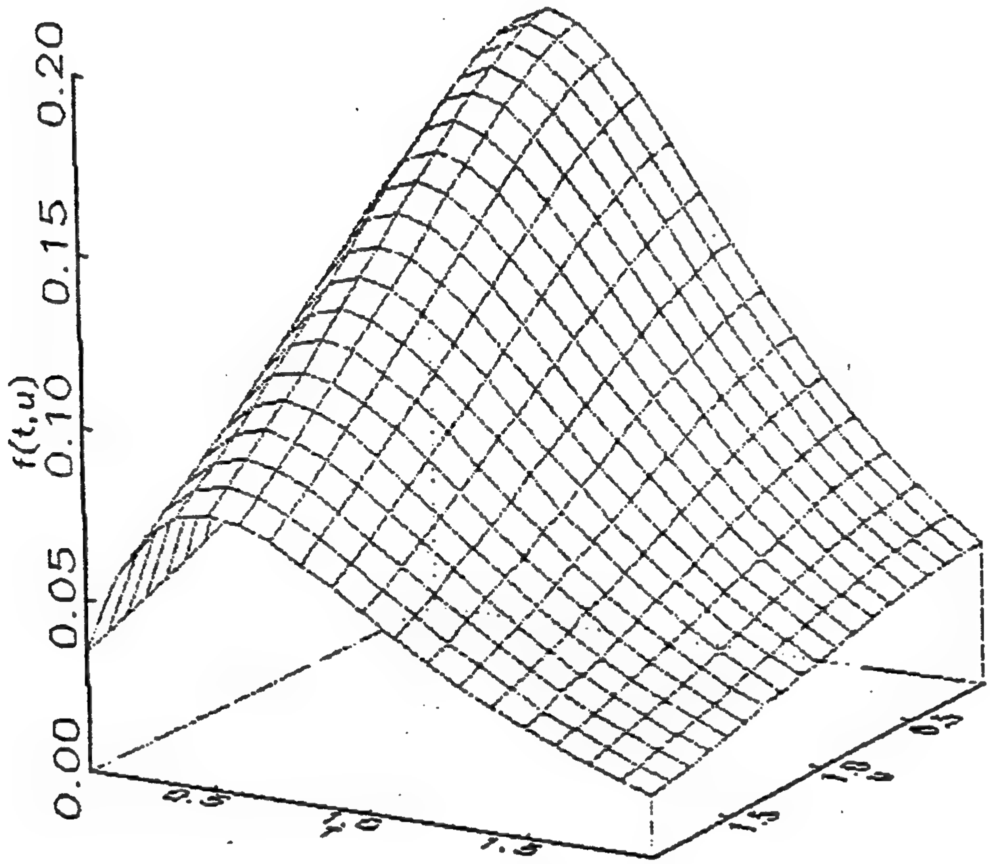

Figure 4 Simulated joint density for (4.7)

5 The usage process

As an aside to the probability model proposed in this thesis, an idealised stochastic process for modelling usage of an automobile, computer or any device where usage is a real-valued quantity is introduced. This process will be called the usage process. The usage process is a function of three sets of non-negative valued random variables :

i) $X_1, X_2, X_3, \ldots$ define the length of consecutive periods of rest and

ii) $Y_1, Y_2, Y_3, \ldots$ define the length of consecutive periods of use and

iii) $G_1, G_2, G_3, \ldots$ define the rate of use during each period of use Y_1, Y_2 etc.

The process starts with rest period X_1. At the end of this period the first usage period Y_1 starts, with usage accumulating at the random rate G_1. When Y_1 finishes, X_2 begins and so on, with each rest period X_i followed by a use period Y_i. During the ith usage period, amount of use increases at a rate G_i. During rest periods the amount of usage stays constant. The amount of use in the i-th use period will be Y_iG_i. Figure 5 shows a typical realisation of the usage process.

Let M(t) denote the amount of use at time t. First of all, if we suppose that at time t we are in the (n+1)th rest period, then usage to time t is given by

$$(5.1) \qquad M(t) = \sum_{i=1}^{n} Y_iG_i \ .$$

Now suppose at time t we are in the (n+1)th use period, which started at time T_{n+1}. Then the usage to time t is given by the following equation :

$$(5.2) \qquad M(t) = \sum_{i=1}^{n} Y_iG_i + (t - T_{n+1})G_{n+1}$$

$$\text{where } T_1 = X_1$$

$$T_i = T_{i-1} + Y_{i-1} + X_i \qquad i=2, 3, \ldots$$

We can combine (5.1) and (5.2) into one expression by

$$M(t) = \sum_{i=1}^{\infty} \min\{ Y_i,\ h(t-T_i)\times(t-T_i) \}\ G_i \tag{5.3}$$

where T_i are defined as in (5.2)

and $h(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

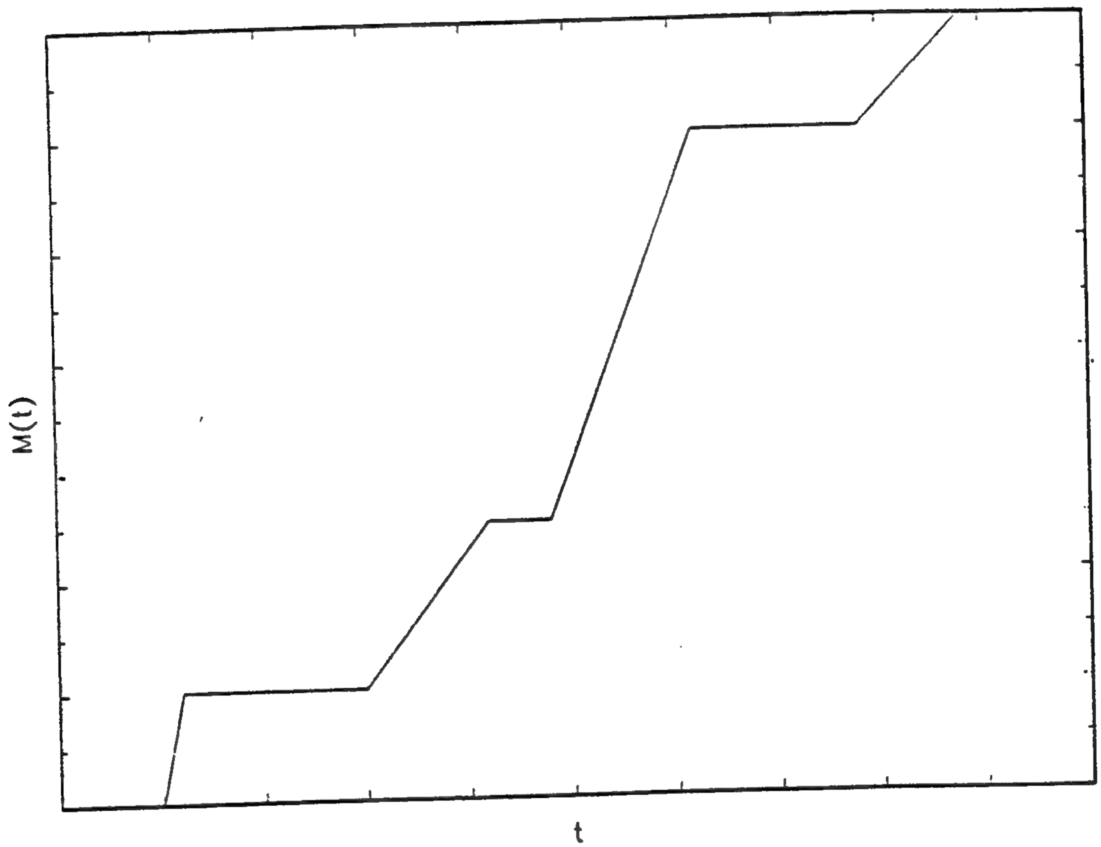

Figure 5 Typical realisation of the usage process M(t).

In practice, one could allow one or more of the three sets of random variables to be deterministic if the situation required it; i.e if M(t) measured the length of time a light bulb is switched on up to time t then $G_i=1\ \forall i$ would be sensible. If the X_i are independent and identically distributed, and the Y_i are too, then the sequence $(X_1, Y_1, X_2, Y_2, \ldots)$ is an alternating renewal process.

6 Conclusion

This thesis is an initial attempt at addressing some of the requirements for models indexed by time and usage. The approach has been to make assumptions about the relationship between the two scales of measurement and to develop a probability model from those assumptions. In section 2, as well as obtaining a class of joint densities, a class of marginal densities for time to failure was obtained that may be considered a viable univariate model for failure in its own right. By allowing specification of a rate of use function $\lambda(t)$ and a hazard function $r(t)$ to describe environmental effects, the model offers great flexibility. The discrete nature of the usage variable U is not, I believe, a cause for concern since one is at liberty to make the unit of use relatively small (so for example in the case of automobile mileage, setting U to be the number of miles driven).

Section 4 considered the replacement of the Poisson process to model usage by some other stochastic process, and gave an example which showed that one can expect greater problems in deriving the model. Generally, my attempts to make usage a continuous quantity have met with only partial success so far.

Section 5 considered a more realistic stochastic process to model usage than is used in the rest of the thesis, named the usage process. Ideally one would like to obtain a model of the form proposed here that used this process for $M(t)$, but there remains a great deal of investigation to be done before that can be achieved. It is quite possible that analytic results will be unobtainable, although one should be able to simulate the process if this is the case. It is my intention to continue research on this particular topic.

In summary, then, this is a new area of research and there are many avenues of thought still to be explored. There are entirely different methods of creating models indexed by two variables that need to be investigated, and the method used here needs to be extended to other usage processes, perhaps to other forms for the conditional failure rate and even to looking at models indexed by three or more dependent variables. In the immediate future, my research in this area will concentrate on the usage process and on trying to develop a model around it.

Appendix

Deriving a failure model indexed by time and usage

This appendix shows how the failure model indexed by time and usage may be derived. It will consider the specific cases that were introduced in sections 2 and 4.

Recall from equation 4.3 that we have a general expression for the density of failure at time t and usage u, when the conditional failure rate is $r(t)+\beta M(t)$ and $M(t)$ is a stochastic process describing usage.

$$\text{(A1)} \quad f(t, u) = f(M(t)=u) \times (r(t)+\beta u)\, e^{-R(t)} \exp\left(-\beta \int_0^t M(s)\, ds \mid M(t)=u\right).$$

In the following two sections we will show the derivation of (A1) when $M(t)$ is a Poisson process (as in section 2) and a compound homogeneous Poisson process (as in section 4).

The Poisson process

Let $M(t)$ be a Poisson process with intensity function $\lambda(t)$. Its density is Poisson with mean $\Lambda(t)$. What about the integral that equation (A1) says we must specify to find the density? Let S_i denote the time of the i-th event in the process. If $M(t)=u$ then the area under the process from 0 to t is :

$$\left(\int_0^t M(s)\, ds \mid M(t)=u\right) = tu - \sum_{i=1}^{u} S_i$$

Conditional on $M(t)=u$ then, as an unordered set, standard Poisson process theory tells us that the S_i's are distributed as a random variable S which has density $\lambda(s)/\Lambda(t)$ on [0, t), and is 0 otherwise. So

$$\text{(A2)} \quad \exp\left(-\beta \int_0^t M(s)\, ds \mid M(t)=u\right) = E\left(\exp\left(-\beta\left(tu-\sum_{i=0}^{u} S_i\right)\right) \mid M(t)=u\right)$$

$$= E\left(\exp\left(-\beta(tu-Su)\right) \mid M(t)=u\right)$$

$$= \left\{ E\left(\exp(-\beta(t-S)) \mid M(t)=u \right) \right\}^u$$

$$= \left\{ \int_0^t \exp(-\beta(t-s)) \cdot \lambda(s)/\Lambda(t) \, ds \right\}^u$$

Substitute this expression into (A1) to obtain equation (2.4) as required.

Compound Poisson Process

In section 4 we attempted to generalise the model of section 2. To simplify matters a little, we assumed that r(t), the non-use failure rate, was a constant α.

Let M(t) be a homogeneous compound Poisson process with intensity λ, and let each jump in the process be an independent and identically distributed exponential random variable with mean 1. Let N(t) be the number of events to time t (so that N(t) is Poisson with mean λt). Then M(t) can be defined as

$$M(t) = \begin{cases} 0, \text{ if } N(t)=0 \\ \sum_{i=1}^{N(t)} U_i \text{ , if } N(t) \geq 1 \end{cases}$$

where the U_i are iid exponential random variables with mean 1.

The density for M(t) can be calculated by conditioning on N(t); if N(t)=n then M(t) is the sum of n iid exponentials with mean 1, so has a gamma distribution with shape n and scale 1. M(t) can take any value in $[0, \infty)$. First note that

$$P(M(t)=0) = P(N(t)=0) = e^{-\lambda t}$$

so that there is mass at M(t)=0. For M(t)=u>0 there is a continuous density function

$$\text{(A3)} \qquad f(M(t)=u) = \sum_{n=1}^{\infty} f(M(t)=u \mid N(t)=n) \, P(N(t)=n)$$

$$= \sum_{n=1}^{\infty} \frac{u^{n-1}e^{-u}}{(n-1)!} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= e^{-u-\lambda t} \sum_{n=1}^{\infty} \frac{u^{n-1}(\lambda t)^n}{(n-1)!\, n!} \;.$$

So for this particular compound Poisson process we have a density function.

As well as the density of M(t), we must specify an exponent of the integral of M(t). This was easy when M(t) was a Poisson process, but now this is more difficult. Let S_i be the epoch of the i-th event, and let U_i be the amount of usage at the i-th event. By observation of figure 3, one can see that the area under the process, conditional on N(t)=n, is

$$\text{(A4)} \qquad \left(\int_0^t M(s)\, ds \mid M(t)=u,\, N(t)=n\right) = \sum_{i=1}^{n} (t-S_i)\, U_i$$

In the Poisson process case M(t)=N(t) but here that is not so. If M(t)=0 then obviously $\int M(s)\, ds=0$ so that f(t| M(t)=0) becomes $\alpha e^{-\alpha t}$. For M(t)=u>0

$$\text{(A5)} \qquad \exp\left\{-\beta \int_0^t M(s)\, ds \mid M(t)=u\right\} = E\left(\exp\left\{-\beta \sum_{i=1}^{N(t)} (t - S_i)\, U_i\right\} \mid M(t)=u\right).$$

Condition on N(t) and average with respect to the distribution of N(t) given M(t)

$$= \sum_{n=1}^{\infty} E\left(\exp\left\{-\beta \sum_{i=1}^{n} (t - S_i)\, U_i\right\} \mid M(t)=u,\, N(t)=n\right) \times P(N(t)=n \mid M(t)=u) \;.$$

Conditional on N(t)=n, then $M(t) = \sum_{i=1}^{n} U_i = u$, so we can say:

$$= \sum_{n=1}^{\infty} E\left(\exp\left\{-\beta t \sum_{i=1}^{n} U_i + \beta \sum_{i=1}^{n} S_i U_i\right\} \mid M(t)=u,\, N(t)=n\right) \times P(N(t)=n \mid M(t)=u)$$

$$= \sum_{n=1}^{\infty} E\left(\exp\left\{-\beta t u + \beta \sum_{i=1}^{n} S_i U_i\right\} \mid M(t)=u,\, N(t)=n\right) \times P(N(t)=n \mid M(t)=u)$$

$$= e^{-\beta t u} \sum_{n=1}^{\infty} E\left(\exp\left\{\beta \sum_{i=1}^{n} S_i U_i\right\} \mid M(t)=u,\, N(t)=n\right) \times P(N(t)=n \mid M(t)=u)$$

..... Equation (A6)

What is $P(N(t)=n| M(t)=u)$, that appears in (A6)? If $u=0$ then $N(t)=0$ with probability 1, so $P(N(t)=0 \mid M(t)=0) = 1$.

If $u>0$ then we have the following :

$$\text{(A7)} \qquad P(N(t)=n| M(t)=u) = \frac{P(N(t)=n)\, P(M(t)=u \mid N(t)=n)}{P(M(t)=u)}$$

$$= \frac{\dfrac{(\lambda t)^n}{n!} e^{-\lambda t} \cdot \dfrac{u^{n-1} e^{-u}}{(n-1)!}}{e^{-\lambda t-u} \sum_{i=1}^{\infty} \dfrac{u^{i-1}(\lambda t)^i}{(i-1)!\, i!}}$$

$$= \frac{\dfrac{u^{n-1}(\lambda t)^n}{(n-1)!\, n!}}{\sum_{i=1}^{\infty} \dfrac{u^{i-1}(\lambda t)^i}{(i-1)!\, i!}} \qquad n=1, 2, 3, \ldots.$$

It can be shown, in a similar fashion to (A7), that conditional on $M(t)=u>0$ and $N(t)=n$ the joint density of $(U_1, \ldots, U_n)$ is

$$\text{(A8)} \qquad f(u_1, \ldots, u_n \mid M(t)=u, N(t)=n) = \frac{(n-1)!}{u^{n-1}} \quad ; \quad \text{for } u_i \geq 0, \sum_{i=1}^{n} u_i = u$$

and that the joint density of $(S_1, \ldots, S_n)$ is

$$\text{(A9)} \qquad f(s_1, \ldots, s_n \mid M(t)=u, N(t)=n) = \frac{n!}{t^n} \quad ; \quad \text{for } 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq t$$

and that the two sets of random variables $(U_1,\ldots,U_n)$ and $(S_1,\ldots,S_n)$ are independent. Use (A7), (A8) and (A9) in equation (A6) to say

$$\exp\left\{-\beta \int_0^t M(s)\, ds \mid M(t)=u\right\}$$

$$= e^{-\beta t u} \sum_{n=1}^{\infty} \frac{\dfrac{u^{n-1}(\lambda t)^n}{(n-1)!\, n!}}{\sum_{i=1}^{\infty} \dfrac{u^{i-1}(\lambda t)^i}{(i-1)!\, i!}} \int_{\Omega_n} \frac{(n-1)!\, n!}{u^{n-1} t^n} \exp \{\beta \sum_{i=1}^{n} s_i u_i\}\, d(\underline{s}, \underline{u})$$

$$\text{(A10)} \qquad = \frac{e^{-\beta t u}}{\sum_{i=1}^{\infty} \frac{u^{i-1}(\lambda t)^i}{(i-1)!\, i!}} \times \sum_{n=1}^{\infty} \lambda^n \int_{\Omega_n} \exp\{\beta \sum_{i=1}^{n} s_i u_i\}\, d(\underset{\sim}{s}, \underset{\sim}{u})$$

where $\Omega_n = \left\{(s_1, ..., s_n, u_1, ..., u_n) \in \mathbf{R}^{2n} \mid u_i \geq 0, \sum_{i=1}^{n} u_i = u,\ 0 \leq s_1 \leq ... \leq s_n \leq t\right\}$

Use (A10) and (A3) in (A1) to give the joint density expression of (4.4) that we require.

Simulation

Numerical integration of (A10) is a forbidding task. Simulation provides a quicker method of approximating (A10), via equations (A5), (A7), (A8) and (A9). A simulation strategy to calculate the density at one point (t, u) is outlined below. Assume α, β and λ known.

Before the strategy is given, there is one observation to make. The distribution of $(U_1,...,U_n)$ given M(t) and N(t), (A7), is that of the order statistics of (n-1) independent uniform random variables on [0,u]. It is easy to show that if we take the order statistics of (n-1) samples from a uniform distribution on [0,u], then the distance between each consecutive order statistic gives a sample $u_1,...,u_n$.

Select (t, u), where u>0 (we know the joint distribution for u=0). The expectation (A5), which forms part of the joint density, is estimated by the mean of a sample of size K, where K is some suitably large number.

Repeat : k=1
- using (A7), sample from the distribution of N(t) | M(t)=u to find a value n^*.

- using (A8), sample from $(U_1, ...,U_{n^*}) \mid M(t)=u,\ N(t)=n^*$. Generate n^*-1 samples from a uniform on [0, u], say $(v_1,...,v_{n^*-1})$. Order them, so that $v_{(1)}<v_{(2)}<...<v_{(n^*-1)}$. Now $u_1=v_{(1)}$, $u_2=v_{(2)}-v_{(1)}$, ..., $u_{n^*}=u-v_{(n^*-1)}$ is the required sample.

- using (A9), sample from $(S_1,\ldots,S_{n^*}) \mid M(t)=u,\ N(t)=n^*$. Generate n^* uniform samples on $[0,t]$. The ordered sample is $s_1, \ldots, s_{n^*}$.

- $e_k = \exp\left\{ \beta \sum_{i=1}^{n^*} s_i u_i \right\}$.

- k=k+1. **Until k=K**

Using (A3) in (A1) we can write the joint density of T and U as

$$f(t, u) = e^{-u-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n\, u^{n-1}}{n!\,(n-1)!} \times (\alpha+\beta u)\, e^{-\alpha t} E\left(\exp\left\{ -\beta \int_0^t M(s)\, ds \right\} \mid M(t)=u \right)$$

$$= (\alpha+\beta u)\, e^{-u-(\alpha+\lambda)t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n\, u^{n-1}}{n!\,(n-1)!} \times E\left(\exp\left\{ -\beta t u + \beta \sum_{i=1}^{N(t)} S_i U_i \right\} \mid M(t)=u \right).$$

Now use the mean of $e_1,\ldots,e_K$ to estimate the expectation that occurs in the above expression:

$$\text{(A11)} \quad f(t, u) \simeq (\alpha+\beta u)\, e^{-u-(\alpha+\lambda)t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n\, u^{n-1}}{n!\,(n-1)!} \times e^{-\beta t u} \frac{1}{K} \sum_{j=1}^{K} e_k .$$

All the three sets of samples to be made - n^*, $(u_1,\ldots,u_{n^*})$ and $(s_1,\ldots,s_{n^*})$ - are easy; the first is a discrete distribution and the latter two only require sampling from a uniform distribution followed by some sorting. Figure 5 gave a plot of a simulated density when $\alpha=\beta=\lambda=1$ using the above procedure. In this case $P(U=0)=0.5$, so the volume under the jointly continuous part of the density is only 0.5. For this simulation $K=10000$ and the infinite sum present in (A11) was truncated at $n=100$ (the sum converges very rapidly). Calculations on a PC took around 4 seconds per density value.

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